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Bidomain Model Solution Using the Finite Volume Method

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Abstract

In this paper we consider cardiac electrical activity through bidomain model, to describe the electrical behavior of cardiac tissue, based on current flow, electric potential distribution and conservation of charge. So we use the finite volume scheme built on rectangular meshes. Discretizing will focus on existing algorithms for elliptic and parabolic equations, with convergence guaranteed by the classical theory.

Keywords: Bidomain Model, finite volumes method, cardiac electrical activity

1 Introduction

Electrical behavior of cardiac tissue can be represented using bidomain model. The tissue is assumed to be composed by two domains, intracellular and extracellular, intertwined and superposed, considered continuous and occupying the entire heart volume, the two are separated by the cardiac cell membrane acting as isolator, the model is based on an approach of average volume equivalent to

a model of multidimensional cable, describing the average electric potentials and flows of intracellular and extracellular current of cardiac muscle [2, 8, 9]. The model consists of a system of partial differential equations of parabolic-elliptic type with Neumann boundary conditions for the resolution of which, we require computationally effective techniques and for this we apply a finite volume scheme. One feature making the method attractive when modeling problems for which flow is important is that of local conservation allowing flow to spray from one cell to another.

The heart is composed or made up on too large a number of cells and so it is not possible to model each cell separately and have full control of potential variations [3, 8, 10, 11]. The bidomain model is based on an approach of average volume, describing the electrical behavior of tissue charge and current conservation [2, 3, 8, 11]. The heart tissue is composed of two domains intra and extra cell domain intertwine and superposed, assumed to be continuos and occupying the entire heart volume, the two are separated by cardiac cell membrane acting as isolator. Spatial domain of our model is a bounded subset $\Omega \subset \mathbb{R}^2$, with smooth boundary $\partial\Omega$. Potentials are denoted by $u_i = u_i(x,t)$ and $u_e = u_e(x,t)$, respectively, $v = v(x,t) := u_i - u_e$ is known as transmembrane potential, the conductivity of the intracellular and extracellular tissue is represented by M_i and M_e, [3, 5, 10, 12]. Using Maxwell equations, the relationship of electric and magnetic fields is given by $\nabla \times E + \frac{\partial B}{\partial t} = 0$, where E and B are the electric and magnetic fields, respectively. Current J is then given by $J = ME$ and $J = -M\nabla u$, where M is conductivity, so $J_i = -M_i \nabla u_i$, $J_e = -M_i \nabla u_e$. Let us assume that the only path of current flow is through the cell membrane $i_{app}(x,t) = i_{app}$ is a stimuli current applied to extracellular space. Now intracellular current through the membrane per unit volume can be expressed as $\nabla \cdot J_i = -I_m$, $\nabla \cdot J_e = I_m - i_{app}$ and $I_m = \beta \left[C_m \frac{\partial v}{\partial t} + I_{ion} \right]$ where I_m is the transmembrane current per unit volume composed of capacity and ionic current (see, for example, [1, 2, 5, 8, 10]). Also, $\nabla \cdot M_i \nabla u_i = I_m$, $\nabla \cdot M_e \nabla u_e - i_{app} = -I_m$ and $-\nabla \cdot M_i \nabla u_i = \nabla \cdot M_e \nabla u_e + i_{app}$. Moreover, by the law of current conservation $\nabla \cdot (M_i \nabla u_i) = -\nabla \cdot (M_e \nabla u_e)$ and using these relations, we obtain

$$\beta C_m \frac{\partial v}{\partial t} - \nabla \cdot (M_i \nabla u_i) + \beta I_{ion} = 0$$
$$\beta C_m \frac{\partial v}{\partial t} + \nabla \cdot (M_e \nabla u_e) + \beta I_{ion} = i_{app}.$$

Now considering Fitzhugh- Nagumo model [4, 7] given by

$$H(v,w) = av - bw \quad \text{and} \quad I_{ion}(v,w) = -\lambda(w - v(1-v)(v-\theta))$$

where a, b, λ and θ are given system parameters. Bidomain model is now given

by the following reaction-diffusion system [1, 3, 12]

$$\begin{aligned} \beta C_m \frac{\partial v}{\partial t} - \nabla \cdot (M_i \nabla u_i) + \beta I_{ion} &= 0 \\ \beta C_m \frac{\partial v}{\partial t} + \nabla \cdot (M_e \nabla u_e) + \beta I_{ion} &= i_{app} \\ \frac{\partial v}{\partial t} - H(v, w) &= 0. \end{aligned} \tag{1}$$

Since we are considering cardiac tissue isolated, we will use the condition of null boundary flow, that is to say,

$$(M_j(x)\nabla(u_j)) \cdot n = 0 \quad \text{on} \quad \Gamma := \partial\Omega \times (0, T),\ j \in \{e, i\}. \tag{2}$$

Rewriting (1) in terms of v and u_e, we have that

$$\nabla \cdot (M_i + M_e)\nabla u_e + \nabla \cdot M_i \nabla v = i_{app}. \tag{3}$$

Therefore, the model in terms of v and u_e is strongly coupled parabolic-elliptic system and

$$\begin{aligned} \beta C_m \frac{\partial v}{\partial t} + \nabla \cdot (M_e \nabla u_e) + \beta I_{ion} &= i_{app} \\ \nabla \cdot (M_i + M_e)\nabla u_e + \nabla \cdot M_i \nabla v &= i_{app} \\ \frac{\partial v}{\partial t} - H(v, w) &= 0. \end{aligned} \tag{4}$$

We impose initial conditions $v(0, x) = v_0(x)$, $w(0, x) = w_0$, $x \in \Omega$.

To find solution we require that v in the bidomain model that initial value v_0 be compatible with (2), so that, if u_i and u_e are fixed values [3], the problem may have no solution, which leads us to impose compatibility condition

$$\int_\Omega u_e(x, t)dx = 0 \quad \text{for almost everywhere}, t \in (0, T).$$

2 Applying the finite volume scheme to bidomain Model

Let a mesh be determined by a family of control volume τ, composed of open rectangles of maximum diameter h, also for all $k \in \tau$, x_k denotes its center, $\varepsilon(k)$ the set of all $k-$borders, ε_{int} corresponds to borders inside Ω and ε_{ext} the set of $k-$borders on $\partial\Omega$, then $\varepsilon(k) = \varepsilon_{int}(k) \cup \varepsilon_{ext}(k) \quad \varepsilon_{int}(k) \cap \varepsilon_{ext}(k) = \emptyset \quad \forall k \in \tau$. Given a finite volume k (see e.g. [1, 6]), $N(k)$ is the set of $k-$neighborhoods with common border in k, $d(k, l)$ denoted distance from x_k and x_l, $\sigma = k|l$ is

the segment from k to l and $n_{k,\sigma}$ is the normal unit vector to $\sigma = k|l$ oriented from k to l for all $k \in \tau$; $|k|$ is the measure of cell k.

We have that $\overline{\Omega} = \bigcup_{k\in\tau} \overline{k}$, for $k \cap l = \emptyset$. If $k, l \in \tau$ and $k \neq l$, segments $\overline{x_k x_l}$ and $\sigma = k|l$ are orthogonal the mesh also satisfied condition for some $\alpha > 0$, $\min\left\{\frac{d(k,l)}{dim(k)}\right\} \geq \alpha$ para $k \in \tau$, $l \in N(k)$. To discretize bidomain equations we choose Ω_τ consisting of mesh Ω and time step $\Delta t \geq 0$, and we set $t^{n+1} = t^n + \Delta t$. Now we consider bidomain model (4), integrating each equation over each cell k and time interval $(t^n, t^n + \Delta t)$ gives

$$\beta \int_{t^n}^{t^{n+1}} \int_k C_m \frac{\partial v}{\partial t} dxdt + \int_{t^n}^{t^{n+1}} \int_{\partial\Omega} M_e(x)\nabla u_e \cdot n_{k,l} dxdt + \beta \int_{t^n}^{t^{n+1}} \int_k I_{ion}(v,w) dxdt = \int_{t^n}^{t^{n+1}} \int_k I_{app} dxdt$$

$$\beta \int_{t^n}^{t^{n+1}} \int_{\partial\Omega} (M_i(x)\nabla u_e + M_e(x)\nabla u_e) \cdot n_{k,l} dxdt + \int_{t^n}^{t^{n+1}} \int_{\partial\Omega} M_i(x)\nabla v \cdot n_{K,l} dxdt = \int_{t^n}^{t^{n+1}} \int_K I_{app} dxdt$$

$$\int_{t^n}^{t^{n+1}} \int_k \frac{\partial w}{\partial t} dx = \int_{t^n}^{t^{n+1}} \int_k H(v,w) dx,$$

where the condition of no-flow on borders is taken independently, there are two important properties to point out, divergence theorems and omitting time integration, we assume that diffusive terms are constant in time over each cell.

Now, to discretize each one of terms over each cell $k \in \tau$, we define conductivity tensors as

$$M_{j,k} := \frac{1}{|k|} \int_k M_j(x) dx \quad j \in \{e, i\}. \tag{5}$$

Using (5), we can choose an approximation to diffusive flow bearing in mind this flow is null on external borders σ

$$F_{j,k,l} \approx \begin{cases} \int_\sigma (M_j(x)\nabla u_j) \cdot n_{k,l} dr & \text{if } \sigma \in \varepsilon_{int} \\ 0 & \text{if } \sigma \in \varepsilon_{int}. \end{cases}$$

Hence

$$\int_\sigma (M_j(x)\nabla u_j) \cdot n_{k,l} dr \approx |\sigma_{k,l}| \nabla u_j(y_\sigma) \cdot M_{j,k} n_{k,l}$$

If $\sigma \in \varepsilon_{int(k)}$ define $M_{j,k,\sigma} = |M_{j,k,\sigma} n_{k,\sigma}| \quad j \in \{e, i\}$. So

$$|\sigma_{k,l}| \nabla u_j(y_\sigma) \cdot M_{j,k} n_{k,l} = |\sigma_{k,l}| M_{j,k,\sigma} \nabla u(y_\sigma) \cdot \frac{y_\sigma - x_k}{d(k,\sigma)}.$$

This last equality does not hold in general, only when $M_{j,k,l}n_{k,\sigma} = |M_{j,k,l}n_{k,\sigma}|n_{k,\sigma}$, which actually is the case if $n_{k,\sigma}$ is an eigenvector for $M_{j,k}$. Also recall $d_{k,\sigma}n_{k,\sigma} = y_\sigma - x_k$. So finally the approximation will be given by

$$|\sigma_{k,l}|M_{j,k,\sigma}\nabla u(y_\sigma)\cdot\frac{y_\sigma - x_k}{d(k,\sigma)} \approx |\sigma_{k,l}|M_{j,k,\sigma}\nabla u(y_\sigma)\cdot\frac{u_{j,\sigma}-u_{j,k}}{d(k,\sigma)}.$$

Now, if $x_k \notin \sigma$ the natural expression for F_k, σ would be

$$F_{j,k,l} = |\sigma_{k,l}|M_{j,k,l}\frac{u_{j,\sigma}-u_{j,k}}{d(k,\sigma_{j,k})}. \tag{6}$$

Since the scheme is conservative, we obtain an approximation for $u_{j,\sigma}$. In fact, $-F_{j,k,l} = F_{j,l,k}$ and

$$|\sigma_{k,l}|M_{j,k,l}\frac{u_{j,\sigma}-u_{j,k}}{d(k,\sigma_{k,l})} = \frac{-|\sigma_{k,l}|M_{j,k,l}(u_{j,\sigma}-u_{j,l})}{d(l,\sigma_{k,l})}$$

then

$$u_{j,\sigma} = \frac{M_{j,k,l}u_{j,k}d(l,\sigma_{k,l}) + d(k,\sigma_{k,l})M_{j,l,k}u_{j,l}}{M_{j,k,k}d(l,\sigma_{k,l}) + M_{j,l,k}d(k,\sigma_{k,l})}$$

dividing both up and down by $d(l,\sigma)d(k,\sigma)$,

$$u_{j,\sigma} = \frac{\frac{M_{j,k,l}u_{j,k}}{d(k,\sigma)} + \frac{M_{j,k,l}u_{j,l}}{d(l,\sigma}}{\frac{M_{j,k,l}}{d(k,\sigma)} + \frac{M_{j,l,k}u_{j,l}}{d(l,\sigma)}} \tag{7}$$

where $u_{j,\sigma}$ is the approximation for $u_j(y_\sigma), j \in \{e,i\}$. If $x_k \in \tau$, replacing (7) on (6) and taking $u_\sigma = u_k$, we obtain the value $F_{j,k,l}$ con $\sigma \in \varepsilon_{int}$

$$F_{j,k,l} = \frac{\big|\sigma_{k,l}M_{j,k,l}\big|M_{j,l,k}(u_{j,l}-u_{j,k})}{\big(M_{j,k,l}d(l,\sigma) + M_{j,l,k}d(k,\sigma)\big)}.$$

So the flow on internal borders is $F_{j,k,l} = d^*_{j,k,l}\big|\sigma_{k,l}\big|(u_{j,l}-u_{j,k})$ with $d^*_{k,l} = \frac{M_{j,k,l}M_{j,l,k}}{M_{j,k,l}d(l,\sigma)+M_{j,l,k}d(k,\sigma)}$ border condition comes in, imposing a no-flow condition in borders $d^*_{j,k,l}|\sigma_{k,l}|(u_{j,l}-u_{j,k}) = 0$ for $\sigma \in \varepsilon_{ext}(k)$, $j = \{i,e\}$. Discretizing terms $H(v,w)$ and $I_{ion}(v,w)$ we have $H^{n+1}_k = \frac{1}{\big|k\big|\Delta t}\int_{t^n}^{t^{n+1}}\int_k H(v,w)dxdt$, $I^{n+1}_{ion_k} = \frac{1}{\big|k\big|\Delta t}\int_{t^n}^{t^{n+1}}\int_k I_{ion}(v,w)dxdt$, $I^n_{app_k} = \frac{1}{\big|k\big|\Delta t}\int_{t^n}^{t^{n+1}}\int_k I_{app}(x,t^n)dxdt$, where for simplicity we divided by Δt.

Now the calculi begin with initiation variables

$$v^0_k = \frac{1}{|k|}\int_k v_0(x)dx \quad \text{and} \quad w^0_k = \frac{1}{|k|}\int_k w_0(x)dx$$

and using Euler scheme we have

$$\int_{t^n}^{t^{n+1}} \int_k \frac{\partial v}{\partial t}(x,t)dxdt = |k|\frac{v_k^{n+1}-v_k^n}{\Delta t} \quad \text{and} \quad \int_{t^n}^{t^{n+1}} \int_k \frac{\partial w}{\partial t}(x,t)dxdt = |k|\frac{w_k^{n+1}-w_k^n}{\Delta t}$$

then replacing each discretization we obtain equations

$$\beta C_m|k|\frac{v_k^{n+1}-v_k^n}{\Delta t} + \sum_{\sigma\in\varepsilon_k} F_{e,k,l}^{n+1} + \beta|k|I_{ion_k}^{n+1} = |k|I_{app_k}^{n+1}$$

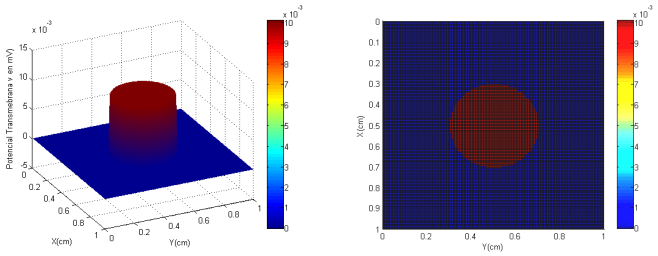
$$\sum_{\sigma\in\varepsilon_k} F_{i,u_e,k,l}^{n+1} + F_{e,u_e,k,l}^{n+1} + F_{i,v,k,l}^{n+1} = |k|I_{app_k}^{n+1} \quad \text{and} \quad \frac{w_k^{n+1}-w_k^n}{\Delta t} - H_k^{n+1} = 0.$$

The final form of finite volume scheme with conditions is now given by

$$\beta C_m|k|\frac{v_k^{n+1}-v_k^n}{\Delta t} + \sum_{\sigma\in\varepsilon_{int}(k)} d_{e,k,l}^*|\sigma_{k,l}|(u_{e,l}^{n+1}-u_{e,k}^{n+1}) + \beta|k|I_{ion_k}^{n+1} = |k|I_{app_k}^{n+1}$$

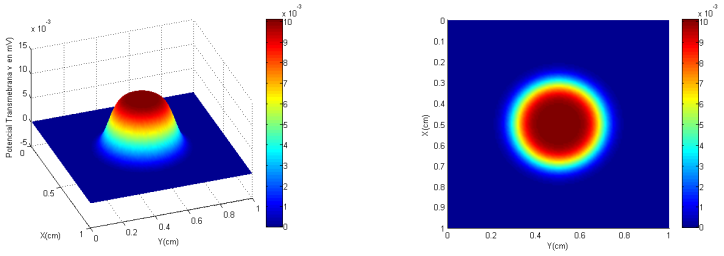
$\sum_{\sigma\in\varepsilon_{int}(k)}|\sigma_{k,l}|d_{i,k,l}^* + d_{e,k,l}^*)(u_{e,l}^{n+1}-u_{e,k}^{n+1}) + d_{i,k,l}^*(v_l^{n+1}-v_k^{n+1}) = |k|I_{app_k}^{n+1}$ and $|k|\frac{w_k^{n+1}-w_k^n}{\Delta t} - |k|H_k^{n+1} = 0$. The border condition is taken imposing a no-flow condition on borders [1, 3] $d_{j,k,l}^*|\sigma_{k,l}|(u_{j,l}^{n+1}-u_{j,k}^{n+1}) = 0$ with $\sigma \in \varepsilon_{ext}(k)$. Respect to boundary conditions of bidomain model, the discretizing is given by $\sum_{k\in\tau}|k|u_{e,k}^{n+1} = 0,\ n = 0,1,2,...,N$.

Finally, we plot result of simulating and programming in Matlab the scheme of finite volumes for the bidomain model. Constants and parameters for this simulation are taken from [1, 7, 13], in such process we consider that myocardial tissue is homogeneous on region $[0,1]\times[0,1]$ and membrane resistance is constant. In the following Figures we observe the variation of initial potential

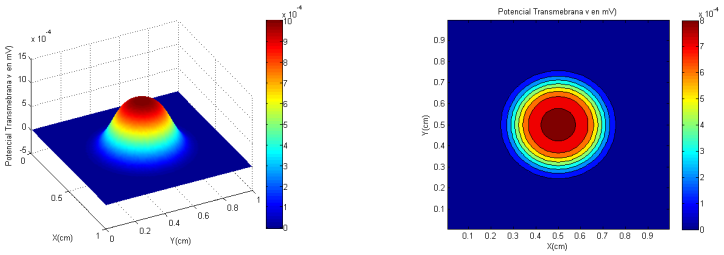


(a) **t=2 ms** (b) **t=2 ms**

$(v = 0)$ when applying an instantaneous stimuli of $1mV$ in $t = 2ms$ to extra cell domain in the miocard in plots c), d). We show the potential evolution in $t = 80ms$. We may observe that evolution v passing the repolarization process, where the values of c) and d) they are in the order of 10^{-3} while e) and f) are 10^{-4}, in said graphic evolution is observed transmembrane potential matching



(c) **t=80ms** (d) **t=80ms**



(e) **t=150ms** (f) **t=150ms**

phases depolarization and repolarization pattern FizHugh- Nagumo, fulfilling the characteristics of a model reaction and diffusion.

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